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Phase representation and its application in the analytical treatment of the theoretical sandpile

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In this paper we introduce a phase representation for sandpile models shown to display self-organized criticality. We find that this phase representation is useful for analyzing these models and for characterizing the evolution of the sandpile. By use of this approach we study the periodic orbits in a version of the deterministic sandpile. The toppling number of each site in every orbit, the period of each orbit, and the number of such orbits are exactly calculated.

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Recently, the behaviors of extended dynamical systems in the far-from-equilibrium state have stimulated much attention and interest. Bak, Tang, and Wiesenfeld (BTW) [1,2] found that such systems naturally evolve into a critical state without tuning any specific parameter such as temperature. BTW define this phenomenon as self-organized criticality (SOC). In BTW's paper they introduced a sandpile automaton to illustrate SOC. In such a model, one at a time particles are stochastically dropped onto one site of a two-dimensional sandpile. If the height or gradient at a certain site exceeds the threshold value assigned to the site, the site topples and the distribution of particles is rearranged according to certain automaton rules. It is found that this model displays SOC in long-time evolution and the final statistically stable critical state is independent of initial conditions. It is much more exciting that this model exhibits the so-called $1/f$ noise by which many phenomena such as sunspot activities, traffic flow, earthquakes, the flow of sand in an hourglass, and the flow of electric current through a resistor are characterized. BTW's concept and model shed much light on the physical origin of the ubiquitous $1/f$ noise in nature. Following BTW's model, many variants of sandpiles were studied mainly by numerical simulation [3–7]. On the other hand, some authors tried to consider this problem in a more mathematical way. Dhar and Rhamaswamy studied the general BTW-type automaton of SOC and described the dynamics of the sandpile using matrix algebra [8,9]. Wiesenfeld, Theiler, and McNamara's simulation and analysis showed that in a deterministic sandpile automaton there exists some periodic orbits whose periods are independent of initial conditions and Dhar's method proved to be a powerful tool in the treatment of this kind of model [10].

In this Rapid Communication we introduce a phase representation. We define some variables such as the state phase and the vector of phase displacement and use them to describe the sandpiles. In such a representation the physical picture of the system is viewed from another point and this can help us understand the sandpile and its evolution more deeply. By use of this approach we studied the periodic attractors in a sandpile and obtained some exact results.

Since our approach is mainly based on Dhar's formalism, we firstly repeat some of his steps and secondly we introduce our concepts and method.

Consider a set of N sites. Define $I=\{1,2,\ldots,N\}$ as the index number set. To each site i is assigned an integer variable Z_i, $i\in I$. This model is called an Abelian model (AM) and it is specified by two rules.

Adding a particle. Select a site i randomly and increase Z_i by 1, leaving other sites unchanged:

$$Z_i \to Z_i+1,$$

$$Z_j \to Z_j \quad (j\neq i)\ .$$

In the language of sandpiles we call this adding a particle to Z_i.

Toppling. The rule is specified by an $N\times N$ integer matrix Δ and a set of N critical values Z_{ic} $(i=1,2,\ldots,N)$. When $Z_i > Z_{ic}$, the site topples and it drops some particles to its neighbors and at the same time some particles may leave the system. If at site i, the toppling occurs,

$$Z_j \to Z_j-\Delta_{ij},\quad \forall j\in I\ ,$$

where Δ_{ij} satisfies

$$\Delta_{ii}>0\ , \tag{1}$$

$$\Delta_{ij}<0,\quad j\neq i,\quad j\in I \tag{2}$$

and

$$\sum_{j=1}^{N}\Delta_{ij}\geq 0,\quad \forall i\in I\ . \tag{3}$$

Equation (3) ensures that there is no creation of particles during the process and the particles can leave the system. It is worth noting that the description of the system and the rules are valid for sandpiles of any dimensions.

We call $\{Z_i|i=1,2,\ldots,N\}$ a configuration of the system and denote it as ψ. Define TS to be the set of all possible configurations of the system. When a configuration

$\psi=\{Z_i\}$ satisfies $Z_i \le Z_{ic}$, $\forall i \in I$, we call this configuration stable. Define S to be the set of all stable configurations. Define N operators a_i, $i=1,2,\ldots,N$. For $\forall\psi\in TS$, $a_i\psi$ denotes the certain stable configuration into which the system evolves when a particle is added to site i. Dhar proved that $[a_i,a_j]=0$, $\forall i,j\in I$. That means that a_i and a_j commute with each other and this is the reason why this model is called the Abelian model. Because the total number of elements in the set S is finite, one can define the recurrent set

$$R_i=\{\psi\in S\,|\,a_i^{m_i}\psi=\psi,\ \exists m_i\in P\}\ ,$$

where P is the set of positive integers. There is still another important property to be noted [11]: for the Abelian model,

$$R_i=R,\quad \forall i\in I\ .$$

This equation means that the recurrent set for every specific site is the same.

Dhar derived that [9] for $\forall\psi\in R$

$$a_i^{\Delta_{ii}}\psi={\prod_j}' a_j^{-\Delta_{ij}}\psi\ ,$$

where the primed product sign indicates the product over all $j\neq i$. We remove ψ from both sides of the equation and get

$$\prod_{j=1}^{N} a_j^{\Delta_{ij}}=1,\quad \forall i\in I\ . \tag{4}$$

Dhar introduced phase ϕ and we can write $a_j=\exp(2\pi i\phi_j)$, $\forall j\in I$. In terms of ϕ's, Eq. (4) can be written as

$$\sum_{j=1}^{N}\Delta_{ij}\phi_j=n_i,\quad \forall i\in I \tag{5}$$

where n_i are some integers. Solving Eq. (5) we get

$$\phi_i=\sum_{j=1}^{N}[\Delta^{-1}]_{ij}n_j,\quad \forall i\in I\ . \tag{6}$$

We define the following.

(i) $\mathbf{Z}=(Z_1,Z_2,\ldots,Z_N)$ the height vector. Z_i stands for the height of the site i, i.e., the number of particles at site i.

(ii) $\boldsymbol{\phi}=(\phi_1,\phi_2,\ldots,\phi_N)$ the phase unit vector. ϕ_i stands for the phase unit corresponding to the operator a_i which satisfies Eq. (4).

(iii) $\Phi=\mathbf{Z}\cdot\boldsymbol{\phi}=\sum_{i=1}^{N}Z_i\phi_i$ the state phase of the system studied.

(iv) $\boldsymbol{l}=\mathbf{Z}_{\text{end}}-\mathbf{Z}_0=(l_1,l_2,\ldots,l_N)$. $\boldsymbol{l}$ stands for the change between an initial and a final configuration, then $\boldsymbol{l}\cdot\boldsymbol{\phi}$ is the difference of the state phase between the initial and the final configuration.

(v) $\mathbf{h}=(h_1,h_2,\ldots,h_N)$ the addition vector. h_i means the added particles at site i from out of the system during a process, then $\mathbf{h}\cdot\boldsymbol{\phi}$ is the phase flowing into the system during a process.

(vi) $\mathbf{s}=(s_1,s_2,\ldots,s_N)$ the toppling vector. s_i stands for the number of topplings at site i during a process.

(vii) $\mathbf{n}=(n_1,n_2,\ldots,n_N)$ the vector of phase displacement. When an $\mathbf{n}$ is specified $\boldsymbol{\phi}$ is determined by Eq. (6), namely, $\boldsymbol{\phi}=\Delta^{-1}\cdot\mathbf{n}$. From Eq. (5) we can see that n_i stands for the reduction of the state phase of the system when there is a toppling at site i. That is the reason why $\mathbf{n}$ is called the vector of phase displacement. In the process of adding→toppling→stable→adding $\cdots$, we view the action of adding a particle at site i as phase ϕ_i flowing into the system and a toppling at site i as phase n_i flowing out of the system. In this picture we can characterize the evolution of the system as the unceasing flowing of phase into or out of the system.

The phase flowing out of the system during a process is $\mathbf{s}\cdot\mathbf{n}$. Hence

$$\mathbf{s}\cdot\mathbf{n}=\mathbf{h}\cdot\boldsymbol{\phi}-\boldsymbol{l}\cdot\boldsymbol{\phi}=(\mathbf{h}-\boldsymbol{l})\cdot\boldsymbol{\phi}\ . \tag{7}$$

Applying Eq. (6) and because of the arbitrary choice of $\mathbf{n}$, we get

$$\mathbf{h}-\boldsymbol{l}=\Delta^{T}\cdot\mathbf{s}\ , \tag{8}$$

where Δ^T is the transposed matrix of Δ. Equation (8) constructs a relationship among the three vectors $\mathbf{h},\boldsymbol{l},\mathbf{s}$.

Now we come back to the recurrent set R defined before. Wiesenfeld, Theiler, and McNamara proved that [10] all the periodic orbits generated by operator a_i have the same period T_i. Based on this property we consider an orbit of a_i. After a period, the change of the state phase of the system is zero, and we write

$$T_i\phi_i-\sum_{j=1}^{N}s_{ij}n_j=0\ . \tag{9}$$

Applying Eq. (6) we write

$$T_i\sum_{j=1}^{N}[\Delta^{-1}]_{ij}n_j-\sum_{j=1}^{N}s_{ij}n_j=0\ ,$$

namely,

$$\sum_{j=1}^{N}(T_i[\Delta^{-1}]_{ij}-s_{ij})n_j=0\ . \tag{10}$$

As for arbitrary $\mathbf{n}$ Eq. (10) holds we get

$$s_{ij}=T_i[\Delta^{-1}]_{ij}\ ,\quad \forall j\in I\ . \tag{11}$$

Thus we get a result: In any two orbits generated by operator a_i, $\forall i\in I$ the toppling number of arbitrary site j is just the same and is determined by Eq. (11). Consequently the total number of topplings during one period in two different orbits is the same.

Let G_{ij} be the probability of toppling at site j when a toppling occurred at site i. From Eq. (11) we write

$$G_{ij}=\frac{s_{ij}}{T_i}=[\Delta^{-1}]_{ij}\ . \tag{12}$$

Equation (12) is identical to Dhar's result [9] and $[\Delta^{-1}]_{ij}$ represents the correlation density between site i and j.

Because Δ_{ij} is an integer matrix, $[\Delta^{-1}]_{ij}$ is a rational matrix. Let $[\Delta^{-1}]_{ij}=N_{ij}/D_{ij}$, where N_{ij} and D_{ij} are prime to each other. Let T_{0i} be the least common multiple of $D_{i1},D_{i2},\ldots,D_{iN}$. Because s_{ij} is an integer the period T_i must be a multiple of T_{0i} according to Eq. (11), namely,

$$T_i = m_i T_{0i}, \quad m_i = 1, 2, \ldots . \tag{13}$$

On a two-dimensional square lattice, under the conditions of free boundary, nearest-neighbor, and isotropic interaction we do a numerical simulation in a lattice of $L \times L$ with $L = 2,3,4$. The free boundary condition means that the heights of sandpiles out of the considered region are always 0. The isotropic interaction means that in an N-dimensional space lattice when a site topples it drops an equal number of particles to its $2N$ nearest neighbors. The simulation reveals that the real period for the lattice is just the minimally possible period T_{0i}, $i = 1,2,\ldots,N = L \times L$ and the data agree with the data obtained in Ref. [10]. Numerical simulation on a one-dimensional lattice also gives the same result. By considering the symmetry of this kind of sandpile model we argue that for any N, the period T_i, $i = 1,2,\ldots,N$ equals T_{0i} defined above.

As the total number of configurations in R is $\det\Delta$ [9], we get the number of periodic orbits corresponding to operator a_i:

$$N_i = \frac{N_R}{T_i} = \frac{\det\Delta}{T_{0i}} . \tag{14}$$

Summary. In this Rapid Communication we introduce the phase representation to the treatment of theoretical sandpiles. In this representation, we studied the periodic orbits of a type of deterministic sandpile and exactly calculated every site's toppling number, the orbits' periods, and the number of the orbits. These results appear in Eqs. (11), (13), and (14). This shows that this method is useful in the analysis of such problems. The results we obtained give us more information about the sandpile and a clearer picture of the sandpile's evolution.

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